$<8$ μs. In the $C$-band, a high-power variable power divider–combiner is realized by using the construction of case 2 in Fig. 1 with the following performance: bandwidth = 10 percent, insertion loss < 0.6 dB, VSWR < 1.25, switching time < 20 μs, peak power = 500 kW, average power = 500 W, phase shift between two orthogossionnal modes of the quadrupole-field section = 90°, power division–combination ratio = 3 dB.

The above results show that experiments are in good agreement with theory, and some practical devices have been constructed with many advantages over devices with electromagnets and holding currents.

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# Generalized Lorentz Gauge and Boundary Conditions in Partially Dielectric-Loaded Cylindrical Waveguide

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***Abstract*—A generalized Lorentz gauge condition for the set of Vlasov–Maxwell equations is introduced. The condition is applied to the free-electron-laser instability of a relativistic electron beam in a partially dielectric-loaded waveguide. For the dielectric-loaded system with the external wiggler magnetic field, the potential approach with the generalized Lorentz gauge rather than the field approach is shown to be more convenient in the self-consistent study of free-electron-laser instability. We also derive the boundary conditions for potentials to be satisfied at the vacuum–dielectric interface and show that they are equivalent to the $B_\theta$ and $E_\theta$ continuous conditions in the field approach. An example is discussed to illustrate the equivalence between the two approaches of potentials and fields.**

## I. Introduction

The instabilities of electromagnetic waves in dielectric-loaded cylindrical waveguide have been the subject of a number of recent investigations. [1]–[3] These works make use of the fluid Maxwell description and neglect the radial effect of the relativistic electron beam. Using the Vlasov–Maxwell scheme, Uhm and Davidson [4], [5] investigated the properties of free-electron-laser instability in a relativistic electron beam, which has a finite radial profile, propagating through a cylindrical vacuum waveguide. In their problem of the vacuum-beam boundary conditions for the system with the external wiggler magnetic field, the perturbed potential rather than the field was used in the self-consistent calculations of current and charge densities. To formulate the problem self-consistently, the potential approach is more convenient and is used extensively. For the problem of partially dielectric-loaded waveguide extending the vacuum waveguide case, the boundary conditions of potential quantities at the vacuum–dielectric interface are required. In this study the boundary conditions of the scalar and vector potentials at the vacuum–dielectric interface in a dielectric-loaded cylindrical waveguide are presented. In our analysis, we make use of a generalized Lorentz gauge condition for the potentials. It is shown that the boundary conditions on potentials are equivalent to the boundary conditions on electromagnetic fields. In addition, we discuss characteristics of the eigenmode that propagates through a partially dielectric-loaded cylindrical waveguide using the derived new potential boundary conditions.

Manuscript received February 26, 1987; revised July 8, 1987.

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IEEE Log Number 8716911.

## II. Formulation

We consider a partially dielectric-loaded cylindrical waveguide with a grounded conducting wall. The permeability of the dielectric material differs from unity by only a few parts in $10^5$; thus in the Maxwell equations the permeability is set as $\mu = 1$. The displacement vector $\boldsymbol{D}$ is related to $\boldsymbol{E}$ as $\boldsymbol{D} = \epsilon \boldsymbol{E}$, where $\epsilon$ is the dielectric constant. Cylindrical coordinates $(r, \theta, z)$ are introduced and the dielectric constant is assumed to be only a function of the radial variable $r$.

In this analysis, a normal mode approach is adopted in which all quantities are assumed to vary according to

$$\Psi(x, t) = \hat{\Psi}(r) \exp[i(l\theta + kz - \omega t)] \tag{1}$$

where $l$ is the azimuthal harmonic number, $k$ is the axial wavenumber, $\omega$ is the eigenfrequency, and $\hat{\Psi}(r)$ is the amplitude. The scalar potential $\phi$ and the vector potential $\boldsymbol{A}$ are related to the fields $\boldsymbol{B}$ and $\boldsymbol{E}$ as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{2}$$

$$\boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}. \tag{3}$$

Choosing the gauge condition

$$\nabla \cdot \boldsymbol{A} + \frac{\epsilon}{c}\frac{\partial \phi}{\partial t} = 0 \tag{4}$$

as the generalization of the Lorentz gauge to the case with dielectrics, the Maxwell equations for the potentials $A_r$, $A_\theta$, $A_z$, and $\phi$ are given as

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{l^2}{r^2} + p^2\right)\hat{\phi}(r) + \frac{1}{\epsilon}\frac{d\epsilon}{dr}\frac{d\hat{\phi}}{dr} - \frac{i\omega}{\epsilon c}\frac{d\epsilon}{dr}\hat{A}_r(r) = 0 \tag{5}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{l^2+1}{r^2} + p^2\right)\hat{A}_r(r) - \frac{2il}{r^2}\hat{A}_\theta(r) - \frac{i\omega}{c}\hat{\phi}(r)\frac{d\epsilon}{dr} = 0 \tag{6}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{l^2+1}{r^2} + p^2\right)\hat{A}_\theta(r) + \frac{2il}{r^2}\hat{A}_r(r) = 0 \tag{7}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{l^2}{r^2} + p^2\right)\hat{A}_z(r) = 0 \tag{8}$$

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where

$$p^2 = \epsilon\omega^2/c^2 - k^2. \tag{9}$$

A cylindrical waveguide with a dielectric material in the range $R_w < r < R_c$ and a vacuum in $r < R_w$ is considered. The radial profile of the dielectric constant is given by

$$\epsilon(r) = \begin{cases} 1 & \text{for } 0 \leqslant r < R_w \\ \hat{\epsilon} & \text{for } R_w < r \leqslant R_c \end{cases} \tag{10}$$

where $\hat{\epsilon}$ is a constant lager than unity. $R_c$ is the radius of a grounded conducting wall. The boundary conditions of potential quantities at the vacuum–dielectric interface $r = R_w$ are investigated for the cylindrical dielectric-loaded waveguide. Since the field quantities are represented by the derivative of potentials from (2), (3), and (4), the potentials $\hat{\phi}$, $\hat{A}_r$, $\hat{A}_\theta$, and $\hat{A}_z$ should be defined as the continuous functions at $r = R_w$. That is,

$$\hat{\phi}, \hat{A}: \text{continuous at } r = R_w. \tag{11}$$

Equation (5) is rewritten as

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\epsilon r\frac{\partial}{\partial r} - \epsilon\frac{l^2}{r^2} + \epsilon p^2\right)\hat{\phi}(r) = \frac{i\omega}{c}\frac{d\epsilon}{dr}\hat{A}_r(r). \tag{12}$$

Multiplying (6) and (12) by $r$ and integrating from $R_w(1-\delta)$ to $R_w(1+\delta)$, with $\delta \to 0$, we obtain

$$\left(\frac{\partial \hat{A}_r}{\partial r}\right)_{R_w(1+\delta)} - \left(\frac{\partial \hat{A}_r}{\partial r}\right)_{R_w(1-\delta)} = \frac{i\omega}{c}(\hat{\epsilon}-1)\hat{\phi}(R_w) \tag{13}$$

$$\hat{\epsilon}\left(\frac{\partial \hat{\phi}}{\partial r}\right)_{R_w(1+\delta)} - \left(\frac{\partial \hat{\phi}}{\partial r}\right)_{R_w(1-\delta)} = \frac{i\omega}{c}(\hat{\epsilon}-1)\hat{A}_r(R_w). \tag{14}$$

Applying the same method to (7) and (8), we note that $\partial\hat{A}_\theta/\partial r$ and $\partial\hat{A}_z/\partial r$ are continuous at $r = R_w$; that is,

$$\frac{\partial \hat{A}_\theta}{\partial r} \text{ and } \frac{\partial \hat{A}_z}{\partial r}: \text{continuous at } r = R_w. \tag{15}$$

It is shown that $\hat{E}_z$ is continuous at $r = R_w$ from (1) and (3) and $\hat{B}_z$ is continuous at $r = R_w$ from (11), (15), and (2). Using (2), (3), (11), and (15), the Maxwell equation of $p^2\hat{B}_\theta = -\frac{kl}{r}\hat{B}_z + i\frac{\epsilon\omega}{c}\frac{\partial}{\partial r}\hat{E}_z$, frequently used as the $B_\theta$ continuity condition in the works where the field rather than the potential approach is taken, is shown to be equivalent to the equation (14) expressing the discontinuity of $\partial\hat{\phi}/\partial r$ at $r = R_w$. Similarly, using (7), the Maxwell equation $p^2\hat{E}_\theta = -\frac{kl}{r}\hat{E}_z - i\frac{\omega}{c}\frac{\partial}{\partial r}\hat{B}_z$ of the $E_\theta$ continuity condition is equivalent to the discontinuity of $\partial\hat{A}_r/\partial r$, eq. (13), at $r = R_w$. Therefore, we conclude that the generalized Lorentz gauge condition of (4) is proper in view of the fact that the boundary conditions of the two different approaches of field and potential are shown to be equivalent.

As a specific example, we investigate the properties of eigenmodes in a cylindrical dielectric-loaded waveguide with a grounded conducting wall. At $r = R_c$ the boundary conditions of potential quantities become

$$\hat{\phi}(R_c) = \hat{A}_z(R_c) = \hat{A}_\theta(R_c) = \left[\frac{\partial r\hat{A}_r}{\partial r}\right]_{R_c} = 0 \tag{16}$$

where use has been made of (2), (3), (4), (7), and $\hat{E}_z(R_c) = (\partial\hat{B}_z/\partial r)_{r=R_c} = 0$. For azimuthal symmetry ($l = 0$), using the boundary conditions of potential quantities (11), (13), (14), (15), and (16), and solving the Maxwell equations (5)–(8) represented by the potentials, we obtain the dispersion relation of $D_{T,0}^E = 0$ with the condition $\hat{\phi}(r) = \hat{A}_r(r) = \hat{A}_z(r) = 0$ (TE mode) or $D_{T,0}^M = 0$ with $\hat{A}_\theta(r) = \hat{A}_z(r) = 0$ (TM mode), here $D_{T,l}^E$ and $D_{T,l}^M$ are the dispersion functions defined as

$$D_{T,l}^E(\omega,k) = \frac{1}{\eta}\,\frac{J_l'(\eta)N_l'\left(\eta\frac{R_c}{R_w}\right) - J_l'\left(\eta\frac{R_c}{R_w}\right)N_l'(\eta)}{J_l(\eta)N_l'\left(\eta\frac{R_c}{R_w}\right) - J_l'\left(\eta\frac{R_c}{R_w}\right)N_l(\eta)} - \frac{1}{\xi}\frac{J_l'(\xi)}{J_l(\xi)} \tag{17}$$

$$D_{T,l}^M(\omega,k) = \frac{\hat{\epsilon}}{\eta}\,\frac{J_l'(\eta)N_l\left(\eta\frac{R_c}{R_w}\right) - J_l\left(\eta\frac{R_c}{R_w}\right)N_l'(\eta)}{J_l(\eta)N_l\left(\eta\frac{R_c}{R_w}\right) - J_l\left(\eta\frac{R_c}{R_w}\right)N_l(\eta)} - \frac{1}{\xi}\frac{J_l'(\xi)}{J_l(\xi)} \tag{18}$$

where

$$\frac{\eta^2}{R_w^2} = \hat{\epsilon}\frac{\omega^2}{c^2} - k^2$$

and

$$\frac{\xi^2}{R_w^2} = \frac{\omega^2}{c^2} - k^2.$$

For $l \neq 0$, applying the same procedure we obtain the dispersion relation

$$D_{T,l}^E D_{T,l}^M = \frac{l^4}{\eta^4\xi^4}(\eta^2 - \hat{\epsilon}\xi^2)(\eta^2 - \xi^2) \tag{19}$$

which is same as the result of the field approach [7], as well as the properties of the eigenmodes that the axial component of the vector potential is zero. Of course, the eigenmodes do not have the structure of TE mode and TM mode.

In conclusion, from the generalized Lorentz gauge condition we derive the boundary conditions of potentials at the vacuum—dielectric interface in a partially dielectric-loaded cylindrical waveguide. The results can be used to study the free electron laser in a cylindrical partially dielectric-loaded waveguide and other intense microwave generation dielectric-loaded devices when we apply the Vlasov–Maxwell equations using potentials rather than fields.

## Acknowledgment

The authors are grateful to Dr. H. S. Uhm of the Naval Surface Weapons Center, White Oak, Silver Spring, MD 20910, for many fruitful discussions.

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